Proof: $I = \int_0^1 x \cdot \tan^{-1} x \, dx$ (ILATE)

$= \int_0^1 \tan^{-1} x \cdot x \, dx$
$\quad\quad (u) \quad (v)$

Integrating by parts

$= \left\{ \tan^{-1} x \int x \, dx - \int \left[\frac{d}{dx} \tan^{-1} x \int x \, dx \right] dx \right\}_0^1$

$= \left\{ \tan^{-1} x \cdot \frac{x^2}{2} - \int \left[\frac{1}{1+x^2} \cdot \frac{x^2}{2} \right] dx \right\}_0^1$

$= \frac{1}{2} \left\{ x^2 \cdot \tan^{-1} x - \int \frac{x^2}{1+x^2} \, dx \right\}_0^1$ "Adjustment"

$= \frac{1}{2} \left\{ x^2 \tan^{-1} x - \int \frac{1+x^2-1}{1+x^2} \, dx \right\}_0^1$

$= \frac{1}{2} \left\{ x^2 \tan^{-1} x - \int \left[\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2} \right] dx \right\}_0^1$

$= \frac{1}{2} \left\{ x^2 \tan^{-1} x - x + \tan^{-1} x \right\}_0^1$

$= \frac{1}{2} \left\{ 1 \tan^{-1} 1 - 1 + \tan^{-1} 1 - 0 \tan^{-1} 0 + 0 - \tan^{-1} 0 \right\}$

$\therefore I = \frac{1}{2} \left\{ \frac{\pi}{4} - 1 + \frac{\pi}{4} - 0 + 0 - 0 \right\} = \frac{1}{2} \left\{ \frac{\pi}{2} - 1 \right\} = \frac{\pi}{4} - \frac{1}{2}$

$\therefore \int_0^1 x \tan^{-1} x \, dx = \frac{\pi}{4} - \frac{1}{2}$

Q-10 P.T. $\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{(1-x^2)^{3/2}} \, dx = \frac{\pi}{4} - \log\sqrt{2}$ [C.W. Type-1, Prob. (25)]

Proof: $I = \int_0^1 \frac{\sin^{-1} x}{(1-x^2)^{3/2}} \, dx$

Let $\boxed{\sin^{-1} x = \theta}$ $\quad \boxed{\therefore x = \sin\theta}$

Diff. w.r.t. θ

$\therefore \frac{dx}{d\theta} = \cos\theta$ $\quad \boxed{\therefore dx = \cos\theta \, d\theta}$

Consider $(1-x^2)^{3/2} = (1-\sin^2\theta)^{3/2} = (\cos^2\theta)^{3/2} = \cos^3\theta$

When $x=0 \Rightarrow \theta = \sin^{-1}0 = 0$, $x = \frac{1}{\sqrt{2}}$, $\theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

$$\therefore I = \int_0^{\frac{\pi}{4}} \frac{\theta \cdot \cos\theta \, d\theta}{\cos^3\theta} = \int_0^{\frac{\pi}{4}} \frac{\theta}{\cos^2\theta} \, d\theta$$

$$\therefore I = \int_0^{\frac{\pi}{4}} \underset{(u)}{\theta} \cdot \underset{(v)}{\sec^2\theta} \, d\theta \quad (ILATE)$$

Integrating by parts

$$= \left\{ \theta \int \sec^2\theta \, d\theta - \int \left[\frac{d}{d\theta}\theta \int \sec^2\theta \, d\theta \right] d\theta \right\}_0^{\frac{\pi}{4}}$$

$$= \left\{ \theta \cdot \tan\theta - \int \left[(1)(\tan\theta) \right] d\theta \right\}_0^{\pi/4}$$

$$= \left\{ \theta \cdot \tan\theta - \int \tan\theta \, d\theta \right\}_0^{\pi/4}$$

$$= \left\{ \theta \tan\theta - \log|\sec\theta| \right\}_0^{\pi/4}$$

$$= \frac{\pi}{4}\tan\frac{\pi}{4} - \log\left|\sec\frac{\pi}{4}\right| - 0\tan 0 + \log|\sec 0|$$

$$= \frac{\pi}{4}(1) - \log\sqrt{2} - 0 + \log 1 = \frac{\pi}{4} - \log\sqrt{2}$$

$$\therefore \int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1}x}{(1-x^2)^{3/2}} \, dx = \frac{\pi}{4} - \log\sqrt{2}$$

Q-11 P.T. $\int_1^2 \frac{\log x}{x^2} \, dx = \frac{1}{2}(1-\log 2)$ [C.W. Type-1, Prob. (28)]

Proof: $I = \int_1^2 \frac{\log x}{x^2} \, dx$

$$= \int_1^2 \left[\underset{(u)}{\log x} \cdot \underset{(v)}{\frac{1}{x^2}} \right] dx \quad (ILATE)$$

Integrating by parts

$$\therefore I = \left\{ \log x \int \frac{1}{x^2}\,dx - \int \left[\frac{d}{dx}\log x \int \frac{1}{x^2}\,dx \right] dx \right\}_1^2$$

$$= \left\{ \log x\left(-\frac{1}{x}\right) - \int \left[\left(\frac{1}{x}\right)\left(-\frac{1}{x}\right) \right] dx \right\}_1^2$$

$$= \left\{ -\frac{\log x}{x} + \int \frac{1}{x^2}\,dx \right\}_1^2$$

$$= \left\{ -\frac{\log x}{x} - \frac{1}{x} \right\}_1^2$$

$$= \left\{ -\frac{(\log x + 1)}{x} \right\}_1^2 = -\frac{\log 2 + 1}{2} + \frac{\log 1 + 1}{1} = -\frac{\log 2}{2} - \frac{1}{2} + 1$$

$$\therefore I = \frac{1}{2} - \frac{1}{2}\log 2 = \frac{1}{2}(1 - \log 2) \qquad \therefore \int_1^2 \frac{\log x}{x^2}\,dx = \frac{1}{2}(1 - \log 2)$$

Q-12 P. T. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sqrt[3]{\tan x}} = \frac{\pi}{4}$ [C.W. Type-2, Prob. (2)]

Proof: $\displaystyle I = \int_0^{\pi/2} \frac{dx}{1 + \sqrt[3]{\tan x}}$ — (i)

using the property

$\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we get

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sqrt[3]{\tan\left(\frac{\pi}{2} - x\right)}} = \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \sqrt[3]{\cot x}} = \int_0^{\frac{\pi}{2}} \frac{dx}{1 + \frac{1}{\sqrt[3]{\tan x}}}$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\sqrt[3]{\tan x}}{\sqrt[3]{\tan x} + 1}\,dx \quad \text{— (ii)}$$

(i) + (ii) gives

$$2I = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \sqrt[3]{\tan x}}\,dx + \int_0^{\frac{\pi}{2}} \frac{\sqrt[3]{\tan x}}{\sqrt[3]{\tan x} + 1}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{1}{1 + \sqrt[3]{\tan x}} + \frac{\sqrt[3]{\tan x}}{\sqrt[3]{\tan x} + 1} \right] dx = \int_0^{\frac{\pi}{2}} \frac{1 + \sqrt[3]{\tan x}}{1 + \sqrt[3]{\tan x}}\,dx$$

$\therefore 2I = (x)_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ $\quad \therefore I = \frac{\pi}{4}$ $\quad \therefore \int_0^{\frac{\pi}{2}} \frac{dx}{1+\sqrt[3]{\tan x}} = \frac{\pi}{4}$

Q-13 P.T. $\int_0^a \frac{dx}{x+\sqrt{a^2-x^2}} = \frac{\pi}{4}$ [C.W. Type-2, Prob.(4)]

Proof: $I = \int_0^a \frac{dx}{x+\sqrt{a^2-x^2}}$

Let $\boxed{x = a\sin\theta}$

Diff. w.r.t. θ

$\therefore \frac{dx}{d\theta} = a\cos\theta$ $\quad \boxed{\therefore dx = a\cos\theta\, d\theta}$

Consider $\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta$

$\boxed{\therefore \sqrt{a^2-x^2} = a\cos\theta}$

When $x=0 \Rightarrow a\sin\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$

$x=a \Rightarrow a\sin\theta = a \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{a\cos\theta\, d\theta}{a\sin\theta + a\cos\theta} = \int_0^{\frac{\pi}{2}} \frac{\cancel{a}\cos\theta\, d\theta}{\cancel{a}(\sin\theta+\cos\theta)}$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\theta}\, d\theta \quad \text{---(i)}$$

using the property $\int_0^a f(\theta)\, d\theta = \int_0^a f(a-\theta)\, d\theta$, we get

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)+\cos\left(\frac{\pi}{2}-\theta\right)}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta+\sin\theta}\, d\theta \quad \text{---(ii)}$$

(i) + (ii) gives

$$2I = \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sin\theta+\cos\theta}\, d\theta + \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta+\sin\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{\cancel{\cos\theta+\sin\theta}}\, d\theta$$

$\therefore 2I = (\theta)_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 0 = \frac{\pi}{2} \quad \therefore I = \frac{\pi}{4} \quad \therefore \int_0^a \frac{dx}{x+\sqrt{a^2-x^2}} = \frac{\pi}{4}$

Q-14 P.T. $\int_0^1 \log\left(\frac{1}{x}-1\right) dx = 0$ [C.W. Type-2, Prob. (6)]

Proof: $I = \int_0^1 \log\left(\frac{1}{x}-1\right) dx$

$= \int_0^1 \log\left(\frac{1-x}{x}\right) dx$

$\therefore I = \int_0^1 [\log(1-x) - \log x]\, dx$ — (i)

using the property

$\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$, we get

$I = \int_0^1 [\log[1-(1-x)] - \log(1-x)]\, dx$

$\therefore I = \int_0^1 [\log x - \log(1-x)]\, dx$ — (ii)

(i) + (ii) gives

$2I = \int_0^1 [\log(1-x) - \log x]\, dx + \int_0^1 [\log x - \log(1-x)]\, dx$

$= \int_0^1 [\cancel{\log(1-x)} - \log x + \log x - \cancel{\log(1-x)}]\, dx$

$\therefore 2I = \int_0^1 0\, dx = 0 \quad \therefore I = 0 \quad \therefore \int_0^1 \log\left(\frac{1}{x}-1\right) dx = 0$

Q-15 P.T. $\int_0^1 \frac{\log(1+x)}{1+x^2} dx = \frac{\pi}{8}\log 2$ [C.W. Type-2, Prob. (9)]

Q-26 P.T. $\int_0^{\frac{1}{2}} \frac{dx}{(1-2x^2)\sqrt{1-x^2}} = \frac{1}{2}\log(2+\sqrt{3})$ [H.W. Prob. (18)]

Proof: $I = \int_0^{\frac{1}{2}} \frac{dx}{(1-2x^2)\sqrt{1-x^2}}$

Let $\boxed{x = \sin\theta}$ $\boxed{\therefore \sin^{-1}x = \theta}$

Diff. w.r.t. x

$\therefore \frac{1}{\sqrt{1-x^2}} = \frac{d\theta}{dx}$ $\boxed{\therefore \frac{dx}{\sqrt{1-x^2}} = d\theta}$

when $x=0 \Rightarrow \theta = \sin^{-1}0 = 0$; $x = \frac{1}{2} \Rightarrow \theta = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

$$\therefore I = \int_0^{\frac{\pi}{6}} \frac{d\theta}{1-2\sin^2\theta} = \int_0^{\frac{\pi}{6}} \frac{d\theta}{\cos 2\theta} = \int_0^{\frac{\pi}{6}} \sec 2\theta \, d\theta$$

$$\therefore I = \frac{1}{2}\Big[\log|\sec 2\theta + \tan 2\theta|\Big]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\left[\log\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \log|\sec 0 + \tan 0|\right]$$

$$= \frac{1}{2}\left[\log|2+\sqrt{3}| - \log|1+0|\right] = \frac{1}{2}\left[\log|2+\sqrt{3}| - \log 1^{0}\right]$$

$\therefore I = \frac{1}{2}\log(2+\sqrt{3})$ $\therefore \int_0^{\frac{1}{2}} \frac{dx}{(1-2x^2)\sqrt{1-x^2}} = \frac{1}{2}\log(2+\sqrt{3})$

Q-27 P.T. $\int_0^2 \frac{2^x\,dx}{2^x(1+4^x)} = \log_2\left(\frac{4\sqrt{2}}{\sqrt{17}}\right)$ [H.W. Prob. (20)]

Proof: $I = \int_0^2 \frac{2^x\,dx}{2^x\left[1+(2^x)^2\right]}$ $\{\because 4^x = (2^x)^2\}$ u^2 where $u = 2^x$

Let $\boxed{2^x = t}$

Diff. w.r.t. x

$\therefore 2^x \log 2 = \frac{dt}{dx}$ $\boxed{\therefore 2^x\,dx = \frac{dt}{\log 2}}$

when $x=0 \Rightarrow t = 2^0 = 1$; $x=2 \Rightarrow t = 2^2 = 4$

$$\therefore I = \int_1^4 \frac{dt/\log 2}{t(1+t^2)} = \frac{1}{\log 2}\int_1^4 \frac{1}{t(1+t^2)}\,dt$$

$$= \frac{1}{\log 2}\int_1^4 \frac{(1+t^2)-t^2}{t(1+t^2)}\,dt \quad \otimes$$

spliting

$$= \frac{1}{\log 2}\int_1^4 \left[\frac{(1+t^2)}{t(1+t^2)} - \frac{t^2}{t(1+t^2)}\right]dt$$

$$= \frac{1}{\log 2}\left[\int_1^4 \frac{1}{t}\,dt - \frac{1}{2}\int_1^4 \frac{2t}{1+t^2}\,dt\right] \quad \otimes$$

$$= \frac{1}{\log 2}\left\{\left[\log|t|\right]_1^4 - \frac{1}{2}\left[\log|1+t^2|\right]_1^4\right\}$$

$$= \frac{1}{\log 2}\left\{\log|4| - \log|1| - \frac{1}{2}\left[\log|17| - \log|2|\right]\right\}$$

$$= \frac{1}{\log 2}\left\{\log 4 - \frac{1}{2}\log\left(\frac{17}{2}\right)\right\} = \frac{1}{\log 2}\left\{\log 4 - \log\sqrt{\frac{17}{2}}\right\}$$

$$\therefore I = \frac{1}{\log 2}\log\left(\frac{4}{\sqrt{\frac{17}{2}}}\right) = \frac{\log\left(\frac{4\sqrt{2}}{\sqrt{17}}\right)}{\log 2} = \log_2\left(\frac{4\sqrt{2}}{\sqrt{17}}\right) \left\{\because \frac{\log m}{\log n} = \log_n^m\right\}$$

$$\therefore \int_0^2 \frac{2^x\,dx}{2^x(1+4^x)} = \log_2\left(\frac{4\sqrt{2}}{\sqrt{17}}\right)$$

Q-28 P.T. $\int_2^e \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right]dx = e - \frac{2}{\log 2}$ [H.W. Prob. (23)]

Proof: $I = \int_2^e \left[\underbrace{\frac{1}{\log x} - \frac{1}{(\log x)^2}}_{f(\log x)}\right]dx$

Let $\log x = t$ $\quad \therefore \log_e^x = t$

$\therefore x = e^t$

Diff. w.r.t. t

$\therefore \frac{dx}{dt} = e^t$ $\quad \therefore dx = e^t\,dt$

When $x = 2 \Rightarrow t = \log 2$; $x = e \Rightarrow t = \log e = 1$

$$I = \int_{\log 2}^{1} \left[\frac{1}{t} - \frac{1}{t^2} \right] e^t \, dt$$

$$\therefore I = \int_{\log 2}^{1} e^t \left[\frac{1}{t} + \left(-\frac{1}{t^2} \right) \right] dt$$

Here $f(t) = \frac{1}{t}$ $\quad \therefore f'(t) = -\frac{1}{t^2}$

$$\therefore I = \int_{\log 2}^{1} e^t \left[f(t) + f'(t) \right] dt = \left[e^t f(t) \right]_{\log 2}^{1} =$$

$$\therefore I = \left[\frac{e^t}{t} \right]_{\log 2}^{1} = \frac{e^1}{1} - \frac{e^{\log 2}}{\log 2} = e - \frac{2}{\log 2} \quad \{ \because e^{\log p} = p \}$$

$$\therefore \int_{2}^{e} \left[\frac{1}{\log x} - \frac{1}{(\log x)^2} \right] dx = e - \frac{2}{\log 2}$$

Q-29 P.T. $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2} \right) dx = 0$ [H.W. Prob.(28)]

Proof: $I = \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2} \right) dx$

$$= \int_0^1 \tan^{-1}\left[\frac{x-(1-x)}{1+x(1-x)} \right] dx \quad \{ \because x-(1-x) = 2x-1 \}$$

$$\therefore I = \int_0^1 \left[\tan^{-1} x - \tan^{-1}(1-x) \right] dx \quad \text{--- (i)} \quad \left\{ \because \tan^{-1}\left(\frac{A-B}{1+AB} \right) = \tan^{-1} A - \tan^{-1} B \right\}$$

using the property

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx, \text{ we get}$$

$$\therefore I = \int_0^1 \{\tan^{-1}(1-x) - \tan^{-1}[1-(1-x)]\}\, dx$$

$$\therefore I = \int_0^1 [\tan^{-1}(1-x) - \tan^{-1}x]\, dx$$

$$\therefore I = -\int_0^1 [\tan^{-1}x - \tan^{-1}(1-x)]\, dx$$

$\therefore I = -I$ {from (i)}

$\therefore 2I = 0 \quad \therefore I = 0 \quad \therefore \int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx = 0$

Q-30 P. T. $\int_2^7 \frac{x^3}{(9-x)^3+x^3}\, dx = \frac{5}{2}$ [H.W. Prob. (31)]

Proof: $I = \int_2^7 \frac{x^3}{(9-x)^3+x^3}\, dx$ — (i)

Using the property

$\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$, we get

$$I = \int_2^7 \frac{(2+7-x)^3}{[9-(2+7-x)]^3+(2+7-x)^3}\, dx$$

$$\therefore I = \int_2^7 \frac{(9-x)^3}{x^3+(9-x)^3}\, dx \quad \text{— (ii)}$$

(i) + (ii) gives

$$2I = \int_2^7 \frac{x^3}{(9-x)^3+x^3}\, dx + \int_2^7 \frac{(9-x)^3}{x^3+(9-x)^3}\, dx$$

$$= \int_2^7 \left[\frac{x^3}{(9-x)^3+x^3} + \frac{(9-x)^3}{x^3+(9-x)^3}\right] dx = \int_2^7 \frac{1}{1}\, dx$$

$\therefore 2I = (x)_2^7 \quad \therefore 2I = 7-2 \quad \therefore I = \frac{5}{2} \quad \therefore \int_2^7 \frac{x^3\, dx}{(9-x)^3+x^3} = \frac{5}{2}$