(9) $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2$ (10) $\int_0^3 x^2(3-x)^{5/2}\,dx = \frac{16}{77}(3)^{7/2}$

(11) $\int_0^{\pi} x\sin x\cos^2 x\,dx = \frac{\pi}{3}$ (12) $\int_0^{\pi} \frac{x}{1+\sin^2 x}\,dx = \frac{\pi^2}{2\sqrt{2}}$

(13) $\int_0^{\frac{\pi}{2}} \left[2\log(\sin x) - \log(\sin 2x)\right]dx = -\frac{\pi}{2}\log 2$ (14) $\int_0^1 \frac{\log x}{\sqrt{1-x^2}}\,dx = -\frac{\pi}{2}\log 2$

(15) $\int_{-1}^1 |5x-3|\,dx = \frac{34}{5}$ (16) $\int_0^3 x[x]\,dx = \frac{13}{2}$

where $[x]$ is greatest integer function not greater than x

(17) $\int_{-3}^3 \frac{x^3}{9-x^2}\,dx = 0$ (18) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left(\frac{2+\sin x}{2-\sin x}\right)dx = 0$

(19) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^4 x\,dx}{\sin^4 x+\cos^4 x} = \frac{\pi}{2}$ (20) $\int_{-1}^1 \frac{1+x^3}{9-x^2}\,dx = \frac{1}{3}\log 2$

Type-3 : Integral expressed as limit of a sum

Evaluate the following integrals as limit of sum.

(1) $\int_1^3 (3x-4)\,dx$ (2) $\int_0^2 (3x^2-1)\,dx$ (3) $\int_2^3 7^x\,dx$

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(1) 4 (2) 6 (3) $\frac{294}{\log 7}$

(19) $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x\,dx}{2\tan^2 x+5\tan x+1} = \frac{1}{2\sqrt{2}} \log\left(\frac{3+\sqrt{2}}{3-\sqrt{2}}\right)$ · (20) $\int_0^2 \frac{2^x dx}{2^x(1+4^x)} = \log_2\left(\frac{4\sqrt{2}}{\sqrt{17}}\right.$

(21) $\int_0^{\frac{\pi}{2}} x\sin x\,dx = 1$ (22) $\int_1^4 x^2 \cdot \log x\,dx = \frac{64}{3}\log 4 - 7$

· (23) $\int_2^e \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right] dx = e - \frac{2}{\log 2}$ (24) $\int_1^2 \frac{\log x}{x^3} dx = -\frac{\log 2}{8} + \frac{3}{16}$

(25) $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx = \frac{\pi}{4}$ (26) $\int_0^{\frac{\pi}{2}} \frac{\sqrt[3]{\sec x}}{\sqrt[3]{\sec x}+\sqrt[3]{\operatorname{cosec} x}} dx = \frac{\pi}{4}$

(27) $\int_0^{\frac{\pi}{2}} \log(\tan x)\,dx = 0$ · (28) $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right) dx = 0$

(29) $\int_0^4 \frac{dx}{x+\sqrt{16-x^2}} = \frac{\pi}{4}$ (30) $\int_0^4 \frac{(x+5)^{\frac{1}{3}}}{(9-x)^{1/3}+(x+5)^{1/3}} dx = 2$

· (31) $\int_2^7 \frac{x^3}{(9-x)^3+x^3} dx = \frac{5}{2}$ (32) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\operatorname{cosec} x}} = \frac{\pi}{12}$

(33) $\int_{\frac{\pi}{5}}^{\frac{3\pi}{10}} \frac{\sin x}{\sin x+\cos x} dx = \frac{\pi}{20}$ (34) $\int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\,d\theta = \frac{\pi}{8}\log 2$

(35) $\int_0^1 t^2\sqrt{1-t}\,dt = \frac{16}{105}$ · (36) $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx = \frac{\pi^2}{4}$

(37) $\int_0^{\pi} x\sin x\cos^4 x\,dx = \frac{\pi}{5}$ (38) $\int_0^{\frac{\pi}{2}} \log(\sin\theta)\,d\theta = -\frac{\pi}{2}\log 2$

(39) $\int_{-2}^{3} |x-2|\,dx = \frac{17}{2}$

(40) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^3 \cdot \sin^4 x\,dx = 0$

(41) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \log\left(\frac{2-\tan\theta}{2+\tan\theta}\right) d\theta = 0$

(42) $\int_{-1}^{1} \frac{x^2}{1+x^2}\,dx = \frac{4-\pi}{2}$

(43) $\int_{-a}^{a} \frac{x+x^3}{16-x^2}\,dx = 0$

(44) $\int_{-1}^{1} \frac{x^3+2}{\sqrt{x^2+4}}\,dx = 4\log\left(\frac{1+\sqrt{5}}{2}\right)$

(45) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x+\frac{\pi}{4}}{2-\cos 2x}\,dx = \frac{\pi^2}{6\sqrt{3}}$

(46) If $\int_0^a \sqrt{x}\,dx = 2a\int_0^{\pi/2} \sin^3 x\,dx$ then find the value of $\int_a^{a+1} x\,dx$

Evaluate the following by expressing integrals as limit of sum

(47) $\int_0^4 x^2\,dx$ (48) $\int_1^3 x^3\,dx$ (49) $\int_1^2 (2x+5)\,dx$ (50) $\int_0^2 e^x\,dx$

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(46) when $a=0$, $I=\frac{1}{2}$ & when $a=4$, $I=\frac{9}{2}$

(47) $\frac{64}{3}$ (48) 20 (49) 8 (50) e^2-1



$\therefore 2I = (\theta)_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 0 = \frac{\pi}{2} \quad \therefore I = \frac{\pi}{4} \quad \therefore \int_0^a \frac{dx}{x+\sqrt{a^2-x^2}} = \frac{\pi}{4}$

Q-14 P.T. $\int_0^1 \log\left(\frac{1}{x}-1\right) dx = 0$ [C.W. Type-2, Prob. (6)]

Proof: $I = \int_0^1 \log\left(\frac{1}{x}-1\right) dx$

$= \int_0^1 \log\left(\frac{1-x}{x}\right) dx$

$\therefore I = \int_0^1 [\log(1-x) - \log x]\, dx$ — (i)

using the property $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$, we get

$I = \int_0^1 [\log[1-(1-x)] - \log(1-x)]\, dx$

$\therefore I = \int_0^1 [\log x - \log(1-x)]\, dx$ — (ii)

(i) + (ii) gives

$2I = \int_0^1 [\log(1-x) - \log x]\, dx + \int_0^1 [\log x - \log(1-x)]\, dx$

$= \int_0^1 [\cancel{\log(1-x)} - \cancel{\log x} + \cancel{\log x} - \cancel{\log(1-x)}]\, dx$

$\therefore 2I = \int_0^1 0\, dx = 0 \quad \therefore I = 0 \quad \therefore \int_0^1 \log\left(\frac{1}{x}-1\right) dx = 0$

Q-15 P.T. $\int_0^1 \frac{\log(1+x)}{1+x^2} dx = \frac{\pi}{8}\log 2$ [C.W. Type-2, Prob. (9)]

Proof: $I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$

Let $\boxed{x = \tan\theta}$ $\quad \boxed{\therefore \tan^{-1}x = \theta}$

Diff. w.r.t. x

$\therefore \frac{1}{1+x^2} = \frac{d\theta}{dx}$ $\quad \boxed{\therefore \frac{dx}{1+x^2} = d\theta}$

When $x = 0 \Rightarrow \theta = \tan^{-1}0 = 0$; $x = 1 \Rightarrow \theta = \tan^{-1}1 = \frac{\pi}{4}$

$\therefore I = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\,d\theta$ ——— (i)

Using the property

$\int_0^a f(\theta)\,d\theta = \int_0^a f(a-\theta)\,d\theta$, we get

$I = \int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right]d\theta = \int_0^{\frac{\pi}{4}} \log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right]d\theta$ $\quad \left\{\because \tan\left(\frac{\pi}{4}-\theta\right) = \frac{1-\tan\theta}{1+\tan\theta}\right\}$

$= \int_0^{\frac{\pi}{4}} \log\left[\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right]d\theta = \int_0^{\frac{\pi}{4}} \log\left[\frac{2}{1+\tan\theta}\right]d\theta$

$= \int_0^{\frac{\pi}{4}} \left[\log 2 - \log(1+\tan\theta)\right]d\theta$

$\therefore I = \int_0^{\frac{\pi}{4}} \log 2\,d\theta - \int_0^{\frac{\pi}{4}} \log(1+\tan\theta)\,d\theta$

$\therefore I = (\theta\log 2)_0^{\frac{\pi}{4}} - I$ $\quad \{\text{from (i)}\}$

$\therefore 2I = \frac{\pi}{4}\log 2 - 0$ $\quad \therefore I = \frac{\pi}{8}\log 2$ $\quad \therefore \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\log 2$

Q-16 P.T. $\int_0^{\pi} \frac{x}{1+\sin^2 x}\,dx = \frac{\pi^2}{2\sqrt{2}}$ $\quad$ [C.W. Type-2, Prob.(12)]

Proof: $I = \int_0^{\pi} \frac{x}{1+\sin^2 x}\,dx$ ——

using the property

$\int_0^{2a} f(x)\,dx = \int_0^{a} [f(x) + f(2a-x)]\,dx$, we get

Here $f(x) = \frac{x}{1+\sin^2 x}$, $2a = \pi$ $\therefore a = \frac{\pi}{2}$

$\therefore f(2a-x) = f(\pi - x) = \frac{\pi - x}{1+\sin^2(\pi-x)} = \frac{\pi - x}{1+\sin^2 x}$

$$\therefore I = \int_0^{\frac{\pi}{2}} \left[\frac{x}{1+\sin^2 x} + \frac{\pi - x}{1+\sin^2 x}\right] dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cancel{x} + \pi - \cancel{x}}{1+\sin^2 x}\,dx$$

$$\therefore I = \pi \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin^2 x}\,dx$$

$\div$ Nr & Dr. by $\cos^2 x$

$$= \pi \int_0^{\frac{\pi}{2}} \frac{\frac{1}{\cos^2 x}}{\frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}}\,dx = \pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{\sec^2 x + \tan^2 x}\,dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x\,dx}{1+\tan^2 x + \tan^2 x}$$

$$\therefore I = \pi \int_0^{\frac{\pi}{2}} \frac{\sec^2 x\,dx}{2\tan^2 x + 1}$$

$\tan^2 x$ ← u^2 where $u = \tan x$

Let $\boxed{\tan x = t}$

Diff. w.r.t. x

$\therefore \sec^2 x = \frac{dt}{dx}$ $\qquad \boxed{\therefore \sec^2 x\,dx = dt}$

When $x = 0 \Rightarrow t = \tan 0 = 0$; $x = \frac{\pi}{2} \Rightarrow t = \tan\frac{\pi}{2} = \infty$

$$\therefore I = \pi \int_0^{\infty} \frac{dt}{2t^2+1} = \pi \int_0^{\infty} \frac{dt}{(\sqrt{2}t)^2+(1)^2} \left\{ \frac{1}{x^2+a^2} \right\} (5)$$

$$\therefore I = \pi \cdot \frac{1}{\sqrt{2}} \left[\tan^{-1}\left(\frac{\sqrt{2}t}{1}\right) \right]_0^{\infty}$$

$$\therefore I = \frac{\pi}{\sqrt{2}} \left[\tan^{-1}(\infty) - \tan^{-1}(0) \right]$$

$$= \frac{\pi}{\sqrt{2}} \left[\frac{\pi}{2} - 0 \right]$$

$$\therefore I = \frac{\pi^2}{2\sqrt{2}}$$

Q-17 P.T. $\int_0^1 \frac{\log x}{\sqrt{1-x^2}}\, dx = -\frac{\pi}{2}\log 2$ [C.W. Type-2, Prob. (14)]

Proof: $I = \int_0^1 \frac{\log x}{\sqrt{1-x^2}}\, dx$

Let $x = \sin\theta$ $\quad \therefore \sin^{-1} x = \theta$

Diff. w.r.t. x

$$\therefore \frac{1}{\sqrt{1-x^2}} = \frac{d\theta}{dx} \qquad \therefore \frac{dx}{\sqrt{1-x^2}} = d\theta$$

When $x = 0 \Rightarrow \theta = \sin^{-1} 0 = 0$; $x = 1 \Rightarrow \theta = \sin^{-1} 1 = \frac{\pi}{2}$

$$\therefore I = \int_0^{\frac{\pi}{2}} \log(\sin\theta)\, d\theta \quad \text{—— (i)}$$

Using the property

$$\int_0^a f(\theta)\, d\theta = \int_0^a f(a-\theta)\, d\theta, \text{ we get}$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \log\left[\sin\left(\frac{\pi}{2}-\theta\right)\right] d\theta$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \log(\cos\theta)\, d\theta \quad \text{—— (ii)}$$

(i) + (ii) gives

$$\therefore 2I = \int_0^{\frac{\pi}{2}} \log(\sin\theta)\, d\theta + \int_0^{\frac{\pi}{2}} \log(\cos\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\log(\sin\theta) + \log(\cos\theta)\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \log(\sin\theta \cdot \cos\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin 2\theta}{2}\right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\log(\sin 2\theta) - \log 2\right] d\theta$$

$$\therefore 2I = \int_0^{\frac{\pi}{2}} \log(\sin 2\theta)\, d\theta - \int_0^{\frac{\pi}{2}} \underbrace{\log 2}_{k}\, d\theta$$

$$\therefore 2I = I_1 - (\theta \log 2)_0^{\frac{\pi}{2}}$$

$$\therefore 2I = I_1 - \frac{\pi}{2}\log 2 + 0 \qquad \therefore 2I = I_1 - \frac{\pi}{2}\log 2 \quad \text{— (iii)}$$

where $I_1 = \int_0^{\frac{\pi}{2}} \log(\sin 2\theta)\, d\theta$

Let $\boxed{2\theta = t}$ $\qquad \boxed{\therefore \theta = \frac{t}{2}}$

Diff. w.r.t. t

$\therefore \frac{d\theta}{dt} = \frac{1}{2}$ (1) $\qquad \boxed{\therefore d\theta = \frac{dt}{2}}$

When $\theta = 0 \Rightarrow t = 0$; $\theta = \frac{\pi}{2} \Rightarrow t = \pi$

$$\therefore I_1 = \int_0^{\pi} \log(\sin t)\, \frac{dt}{2}$$

$$\therefore I_1 = \frac{1}{2}\int_0^{\pi} \log(\sin t)\, dt$$

using the property
$\int_0^{2a} f(t)\,dt = \int_0^{a} [f(t) + f(2a-t)]\,dt$, we get

$$I_1 = \frac{1}{2}\int_0^{\frac{\pi}{2}} \{\log(\sin t) + \log[\sin(\pi - t)]\}\,dt$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} [\log(\sin t) + \log(\sin t)]\,dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} 2\log(\sin t)\,dt$$

$$\therefore I_1 = \int_0^{\frac{\pi}{2}} \log(\sin t)\,dt$$

$$\therefore I_1 = \int_0^{\frac{\pi}{2}} \log(\sin\theta)\,d\theta \quad \left\{\because \int_a^b f(t)\,dt = \int_a^b f(\theta)\,d\theta\right\}$$

$\therefore I_1 = I$ {from (i)}

$\therefore$ Eqn (iii) becomes

$2I = I - \frac{\pi}{2}\log 2$ $\quad \therefore I = -\frac{\pi}{2}\log 2$ $\quad \therefore \int_0^1 \frac{\log x}{\sqrt{1-x^2}}\,dx = -\frac{\pi}{2}\log 2$

Q-18 P.T. $\int_0^3 x[x]\,dx = \frac{13}{2}$ [C.W. Type-2, Prob. (16)]

where $[x]$ is the greatest integer function not greater than x

Proof: $I = \int_0^3 x[x]\,dx$

using the property
$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ if $a < c < b$, we get

$$I = \int_0^1 x[x]\,dx + \int_1^2 x[x]\,dx + \int_2^3 x[x]\,dx$$

$\because 0 < x < 1$ $\quad \because 1 < x < 2$ $\quad \because 2 < x < 3$

$$= \int_0^1 x(0)\,dx + \int_1^2 x(1)\,dx + \int_2^3 x(2)\,dx$$

$$\therefore I = \int_0^1 0\,dx + \int_1^2 x\,dx + \int_2^3 2x\,dx$$

$$= 0 + \left(\frac{x^2}{2}\right)_1^2 + \left[2\left(\frac{x^2}{2}\right)\right]_2^3$$

$$= 0 + \frac{1}{2}(4-1) + (9-4)$$

$$\therefore I = \frac{3}{2} + 5 = \frac{13}{2} \qquad \therefore \int_0^3 x[x]\,dx = \frac{13}{2}$$

Q-19 P.T. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\,dx = \frac{\pi}{2}$ [C.W. Type-2, Prob. (19)]

Proof: $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\,dx$

Here $f(x) = \frac{\sin^4 x}{\sin^4 x + \cos^4 x}$

Replace x by $-x$

$$\therefore f(-x) = \frac{\sin^4(-x)}{\sin^4(-x) + \cos^4(-x)} = \frac{(-\sin x)^4}{(-\sin x)^4 + (\cos x)^4} = \frac{\sin^4 x}{\sin^4 x + \cos^4 x}$$

$\therefore f(-x) = f(x) \Rightarrow$ it is an <u>even function</u>.

using the property

$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if f is even, we get

$$I = 2\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\,dx \quad \text{---(i)}$$

using the property

$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, we get

$$I = 2\int_0^{\frac{\pi}{2}} \frac{\sin^4\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)}\,dx$$

$$\therefore I = 2\int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\cos^4 x + \sin^4 x}\,dx \quad \text{---(ii)}$$

(i) + (ii) gives

$$2I = 2\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\,dx + 2\int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\cos^4 x + \sin^4 x}\,dx$$

$$\therefore I = \int_0^{\frac{\pi}{2}} \left[\frac{\sin^4 x}{\sin^4 x + \cos^4 x} + \frac{\cos^4 x}{\cos^4 x + \sin^4 x}\right] dx = \int_0^{\frac{\pi}{2}} \frac{\sin^4 x + \cos^4 x}{\sin^4 x + \cos^4 x}\,dx$$

$$\therefore I = (x)_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\therefore \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}\,dx = \frac{\pi}{2}.$$

Q-20 Evaluate the integral $\int_0^2 (3x^2 - 1)\,dx$ as a limit of sum

Solution: Let $\int_0^2 (3x^2-1)\,dx = \int_a^b f(x)\,dx$

Where $a = 0, b = 2$ & $f(x) = 3x^2 - 1$

Here we divide the interval $[0, 2]$ into n equal subintervals, such that $n \to \infty$. Let h be the width of each subinterval

$$\therefore h = \frac{b-a}{n} \qquad \therefore h = \frac{2-0}{n} \qquad \boxed{\therefore nh = 2}$$

$$\because f(x) = 3x^2 - 1$$

Replace x by $a + rh$

$$\therefore f(a+rh) = 3(a+rh)^2 - 1$$

$$= 3(0+rh)^2 - 1$$

$$\therefore f(a+rh) = 3r^2h^2 - 1$$

$$\text{Consider } \sum_{r=1}^{n} [f(a+rh)\cdot h] = \sum_{r=1}^{n} [(3r^2h^2 - 1)h]$$

Q-31 P.T. $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx = \frac{\pi^2}{4}$ [H.W. Prob. (36)]

Proof: $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$

using the property

$\int_0^{2a} f(x)\, dx = \int_0^{a} [f(x) + f(2a-x)]\, dx$, we get

Here $f(x) = \frac{x \sin x}{1+\cos^2 x}$; $2a = \pi$ $\therefore a = \frac{\pi}{2}$

$\therefore f(2a-x) = f(\pi - x) = \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)} = \frac{(\pi - x)\sin x}{1+(-\cos x)^2} = \frac{\pi \sin x - x \sin x}{1+\cos^2 x}$

$\therefore I = \int_0^{\frac{\pi}{2}} \left[\frac{x \sin x}{1+\cos^2 x} + \frac{\pi \sin x - x \sin x}{1+\cos^2 x} \right] dx$

$= \int_0^{\frac{\pi}{2}} \frac{\cancel{x \sin x} + \pi \sin x - \cancel{x \sin x}}{1+\cos^2 x}\, dx$

$\therefore I = \pi \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x}\, dx$

Let $\boxed{\cos x = t}$

Diff. w.r.t. x

$\therefore -\sin x = \frac{dt}{dx}$ $\boxed{\therefore \sin x\, dx = -dt}$

When $x = 0 \Rightarrow t = \cos 0 = 1$; $x = \frac{\pi}{2} \Rightarrow t = \cos\frac{\pi}{2} = 0$

$\therefore I = \pi \int_1^0 \frac{-dt}{1+t^2} = \pi \int_0^1 \frac{dt}{1+t^2}$ $\left\{ \because \int_a^b f(x)\, dx = -\int_a^b f(x)\, dx \right\}$

$= \pi (\tan^{-1} t)_0^1$

$= \pi [\tan^{-1} 1 - \tan^{-1} 0]$

$\therefore I = \pi \left[\frac{\pi}{4} - 0\right] = \frac{\pi^2}{4}$ $\therefore \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx = \frac{\pi^2}{4}$

Q-32 P.T. $\int_{-2}^{3} |x-2|\,dx = \frac{17}{2}$ [H.W. Prob. (39)]

Proof: $I = \int_{-2}^{3} |x-2|\,dx$

$\because |x-2| = x-2$ if $x-2>0$ ie $x>2$

& $|x-2| = 2-x$ if $x-2<0$ ie $x<2$

$\therefore I = \int_{-2}^{2} |x-2|\,dx + \int_{2}^{3} |x-2|\,dx$

$\because -2 < \boxed{x<2}$ $\quad \because \boxed{2<x} < 3$

$= \int_{-2}^{2} (2-x)\,dx + \int_{2}^{3} (x-2)\,dx$

$= \left(2x - \frac{x^2}{2}\right)_{-2}^{2} + \left(\frac{x^2}{2} - 2x\right)_{2}^{3}$

$= 2(2+2) - \frac{1}{2}(4-4) + \frac{1}{2}(9-4) - 2(3-2)$

$= 2(4) - \frac{1}{2}(0) + \frac{1}{2}(5) - 2(1)$

$\therefore I = 8 + \frac{5}{2} - 2 = \frac{5}{2} + 6 = \frac{17}{2}$ $\quad \therefore \int_{-2}^{3} |x-2|\,dx = \frac{17}{2}$

Q-33 P.T. $\int_{-1}^{1} \frac{x^3+2}{\sqrt{x^2+4}}\,dx = 4\log\left(\frac{1+\sqrt{5}}{2}\right)$ [H.W. Prob. (44)]

Proof: $I = \int_{-1}^{1} \frac{x^3+2}{\sqrt{x^2+4}}\,dx$

Here $f(x) = \frac{x^3+2}{\sqrt{x^2+4}}$

Replace x by $-x$

$\therefore f(-x) = \frac{(-x)^3+2}{\sqrt{(-x)^2+4}} = \frac{-x^3+2}{\sqrt{x^2+4}}$

$\therefore f(-x) \neq f(x)$ & $f(-x) \neq -f(x)$

$\therefore$ it is <u>neither even</u> <u>nor odd</u> function

using the property

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a}\left[f(x)+f(-x)\right]dx, \text{ we get}$$

$$I = \int_0^1 \left[\frac{2+x^3}{\sqrt{x^2+4}} + \frac{2-x^3}{\sqrt{x^2+4}}\right]dx$$

$$= \int_0^1 \left[\frac{2+x^3+2-x^3}{\sqrt{x^2+4}}\right]dx = 4\int_0^1 \frac{1}{\sqrt{x^2+4}}\,dx$$

$$= 4\int_0^1 \frac{1}{\sqrt{x^2+2^2}}\,dx \quad \left\{\frac{1}{\sqrt{x^2+a^2}}\right\} \text{(B)}$$

$$= 4\left[\log\left|x+\sqrt{x^2+2^2}\right|\right]_0^1$$

$$= 4\left[\log\left|1+\sqrt{5}\right| - \log\left|1+\sqrt{1+0}\right|\right]$$

$$\therefore I = 4\left[\log\left|1+\sqrt{5}\right| - \log|2|\right] = 4\log\left|\frac{1+\sqrt{5}}{2}\right|$$

$$\therefore \int_{-1}^{1} \frac{x^3+2}{\sqrt{x^2+4}}\,dx = 4\log\left(\frac{1+\sqrt{5}}{2}\right)$$

Q-34 If $\int_0^a \sqrt{x}\,dx = 2a\int_0^{\frac{\pi}{2}} \sin^3 x\,dx$ then find $\int_a^{a+1} x\,dx$ [H.W. Prob. (46)]

Solution: $\because \int_0^a \sqrt{x}\,dx = 2a\int_0^{\frac{\pi}{2}} \sin^3 x\,dx$

$$\therefore \int_0^a x^{\frac{1}{2}}\,dx = \not{2}a\int_0^{\frac{\pi}{2}} \frac{3\sin x - \sin 3x}{2\not{4}}\,dx$$

$$\therefore \left[\frac{x^{3/2}}{3/2}\right]_0^a = \frac{a}{2}\left[-3\cos x + \frac{1}{3}\cos 3x\right]_0^{\frac{\pi}{2}}$$

$$\therefore \frac{2}{3}\left[a^{3/2}-0\right] = \frac{a}{2}\left[-3\cos\frac{\pi}{2} + \frac{1}{3}\cos\frac{3\pi}{2} + 3\cos 0 - \frac{1}{3}\cos 0\right]$$

$$\therefore \frac{2}{3}a\sqrt{a} = \frac{a}{2}\left[-3(0)+\frac{1}{3}(0)+3(1)-\frac{1}{3}(1)\right]$$

$$\therefore \frac{2}{3}a\sqrt{a} = \frac{a}{2}\left[\frac{4}{3}\right]^{2}$$

$$\therefore a\sqrt{a} - 2a = 0$$

$$\therefore a(\sqrt{a} - 2) = 0$$

$$\therefore a = 0 \quad \text{or} \quad \sqrt{a} = 2$$

$$\boxed{\therefore a = 0} \quad \text{or} \quad \boxed{a = 4}$$

When $a = 0$,

$$I = \int_{a}^{a+1} x\,dx \text{ becomes}$$

$$= \int_{0}^{1} x\,dx$$

$$= \left[\frac{x^2}{2}\right]_{0}^{1}$$

$$= \frac{1}{2}(1-0)$$

$$\boxed{\therefore I = \frac{1}{2}}$$

OR

When $a = 4$,

$$I = \int_{a}^{a+1} x\,dx \text{ becomes}$$

$$= \int_{4}^{5} x\,dx$$

$$= \left[\frac{x^2}{2}\right]_{4}^{5}$$

$$= \frac{1}{2}(25-16)$$

$$\boxed{\therefore I = \frac{9}{2}}$$

Q-35 Evaluate $\int_{1}^{3} x^3\,dx$ as a limit of sum.

Solution: Let $I = \int_{1}^{3} x^3\,dx = \int_{a}^{b} f(x)\,dx$, we get

where $a = 1$, $b = 3$ & $f(x) = x^3$

Here we divide $[1,3]$ into n equal subintervals such that $n \to \infty$. Let h be the width of each subinterval

$$\therefore h = \frac{b-a}{n} \quad \therefore h = \frac{3-1}{n} \quad \therefore nh = 2$$

$$\because f(x) = x^3$$

Replace x by $a + rh$