PAIR OF STRAIGHT LINES

Joint equation of pair of lines whose separate equations are $u = 0$ & $v = 0$ is given by $u \cdot v = 0$.

Joint equation of pair of lines passing through the origin is $ax^2 + 2hxy + by^2 = 0$

This is a second degree homogeneous equation in x & y.

Its auxiliary equation is $bm^2 + 2hm + a = 0$, whose roots are m_1 & m_2 which are the slopes of the two lines represented by $ax^2 + 2hxy + by^2 = 0$.

Where, m_1 & $m_2 = \frac{-h \pm \sqrt{h^2 - ab}}{b}$

Also, $m_1 + m_2 = -\frac{2h}{b}$ & $m_1 \cdot m_2 = \frac{a}{b}$

If θ is the acute angle between the lines represented by $ax^2 + 2hxy + by^2 = 0$, then

$$\tan\theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

(a) For the lines to be coincident: $h^2 - ab = 0$

(b) For the lines to be perpendicular: $a + b = 0$

(c) For the lines to be imaginary: $h^2 - ab < 0$

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General second degree equation in x & y:

The combined equation of two lines is a general second degree equation in x & y.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

But every general second degree equation in x and y need not represent a pair of lines.

We consider a Determinant:

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

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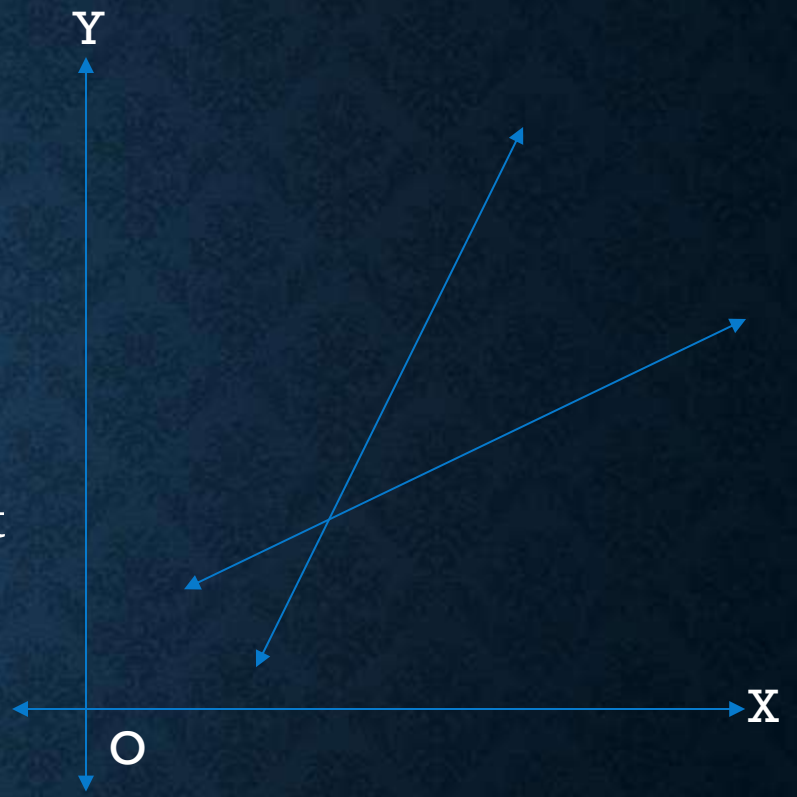
If the value of $D = 0$ & $h^2 - ab \geq 0$, then only the equation represents the joint equation of two lines.



Remarks:

If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines then

1. These lines are parallel to the lines represented by $ax^2 + 2hxy + by^2 = 0$.

2. The acute angle between the lines is given by $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$

   (a) For the lines to be coincident or parallel: $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$, also $h^2 - ab = 0$

   (b) For the lines to be perpendicular: $a + b = 0$

   (c) For the lines to be imaginary: $h^2 - ab < 0$

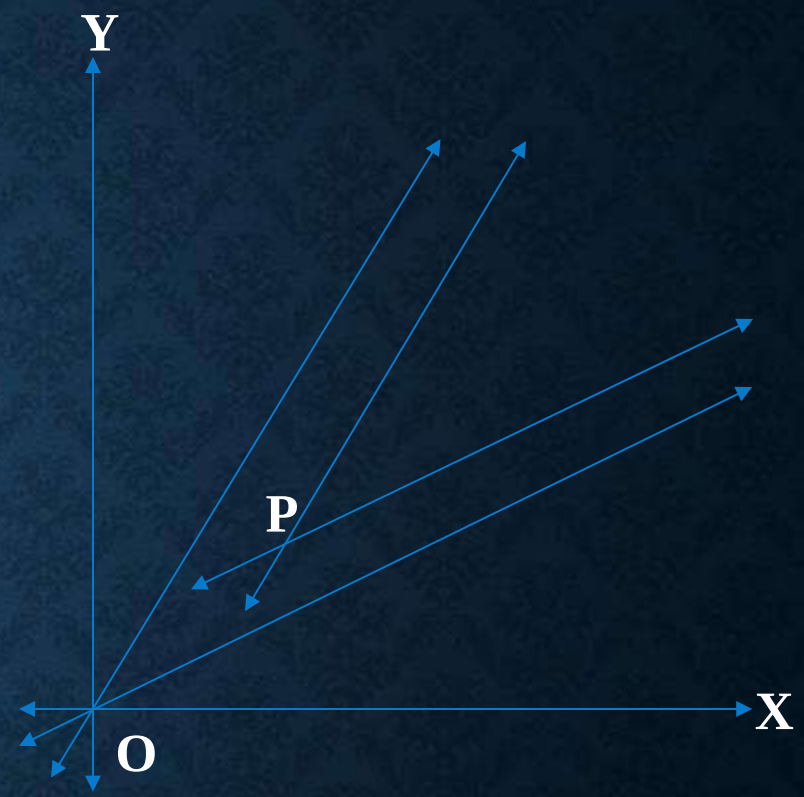
3. Condition for lines to intersect each other is $h^2 - ab > 0$ &

   the co-ordinates of point of intersection are $P \equiv \left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$.

4. The joint equation of bisector of the angle between the lines represented by $ax^2 + 2hxy + by^2 = 0$

   is $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$ or $hx^2 - (a - b)xy - hy^2 = 0$.

   Here coefficient of x^2 + coefficient of $y^2 = 0$, hence bisectors are perpendicular to each other.





Ex. 1. Show that equation $x^2 - 6xy + 5y^2 + 10x - 14y + 9 = 0$ represents a pair of lines. Find the acute angle between them. Also find their point of intersection.

Solution:

The given equation $x^2 - 6xy + 5y^2 + 10x - 14y + 9 = 0$ is a second degree equation in x & y.

Comparing with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get,

$$a = 1, \quad h = -3, \quad b = 5, \quad g = 5, \quad f = -7, \quad c = 9$$

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 1 & -3 & 5 \\ -3 & 5 & -7 \\ 5 & -7 & 9 \end{vmatrix}$$

$$= 1(45 - 49) + 3(-27 + 35) + 5(21 - 25) = -4 + 24 - 20 = 0 \quad \Rightarrow \quad D = 0$$

Also, $h^2 - ab = 9 - 5 = 4 > 0$.

As $D = 0$ and $h^2 - ab > 0$.

Hence the given equation represents a pair of lines.

Let θ be the acute angle between the two lines.

$$\tan\theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right| = \left| \frac{2\sqrt{4}}{1 + 5} \right| = \frac{4}{6} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{2}{3}\right)$$

Their point of intersection is given by

$$\equiv \left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2} \right)$$

$$\equiv \left(\frac{21 - 25}{5 - 9}, \frac{-15 + 7}{5 - 9} \right)$$

$$\equiv (1, 2)$$

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Ex. 2. Find the value of k if the equation $2x^2 + 4xy - 2y^2 + 4x + 8y + k = 0$ represents a pair of lines.

Solution:

The given equation $2x^2 + 4xy - 2y^2 + 4x + 8y + k = 0$ is a second degree equation in x & y.

Comparing with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get,

$a = 2, \quad h = 2, \quad b = -2, \quad g = 2, \quad f = 4, \quad c = k$

As given equation represents a pair of lines,

$\therefore$ It satisfies the condition that $D = 0$.

$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\therefore (2)(-2)(k) + 2(4)(2)(2) - 2(4)^2 - (-2)(2)^2 - (k)(2)^2 = 0$

$\therefore -4k + 32 - 32 + 8 - 4k = 0$

$\therefore 8k = 8$

$\therefore k = 1.$

Ex. 3. Find p and q if the equation $2x^2 + 4xy - py^2 + 4x + qy + 1 = 0$ represents a pair of perpendicular lines.

Solution:

The given equation $2x^2 + 4xy - py^2 + 4x + qy + 1 = 0$ is a second degree equation in x & y.

Comparing with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get,

$a = 2, \quad h = 2, \quad b = -p, \quad g = 2, \quad f = \frac{q}{2}, \quad c = 1$

As the lines are perpendicular to each other, $a + b = 0$

$\therefore 2 + (-p) = 0 \quad \Rightarrow \quad p = 2$

As given equation represents a pair of lines,

$\therefore$ It satisfies the condition that $D = 0$.

$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\therefore -2p + 4q - \frac{q^2}{2} + 4p - 4 = 0 \quad$ { Substituting $p = 2$ }

$\therefore -4 + 4q - \frac{q^2}{2} + 8 - 4 = 0$

$\therefore 8q - q^2 = 0 \quad \Rightarrow \quad q(8 - q) = 0 \quad \Rightarrow \quad q = 0$ or $q = 8$.

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Ex. 4. Find p & q if the equation $2x^2 + 8xy + py^2 + qx + 2y - 15 = 0$ represents a pair of parallel lines.

Solution:

The given equation $2x^2 + 8xy + py^2 + qx + 2y - 15 = 0$ is a second degree equation in x & y.

Comparing with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get,

$$a = 2, \quad h = 4, \quad b = p, \quad g = \frac{q}{2}, \quad f = 1, \quad c = -15$$

As the lines are parallel to each other,

$$\therefore h^2 - ab = 0$$

$$\therefore (4)^2 - (2)(p) = 0 \quad \Rightarrow \quad p = 8$$

Also $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$

$$\therefore \frac{4}{8} = \frac{q}{(2)(1)} \quad \Rightarrow \quad q = 1$$

Ex. 5. Find the joint equation of the pair of lines:

(i) through the point (2, - 1) and parallel to the lines represented by $2x^2 + 3xy - 9y^2 = 0$

(ii) through the point (2, - 3) and parallel to the lines represented by $x^2 + xy - y^2 = 0$.

Solution:

(i) The given joint equation is $2x^2 + 3xy - 9y^2 = 0$

$\therefore 2x^2 + 6xy - 3xy - 9y^2 = 0$

$\therefore 2x(x + 3y) - 3y(x + 3y) = 0$

$\therefore (x + 3y)(2x - 3y) = 0$

$\therefore$ The separate equations of the lines are

$\therefore x + 3y = 0$ **and** $2x - 3y = 0$

$\therefore$ Slopes of the lines are $-\frac{1}{3}$ **and** $\frac{2}{3}$ **.**

$\because$ The required lines are parallel to the given lines.

$\therefore$ Slopes of the required lines are $-\frac{1}{3}$ **and** $\frac{2}{3}$ **.**

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As the lines are passing through the point (2, – 1)

∴ The equation of the required lines by slope point are:

$\therefore y + 1 = -\frac{1}{3}(x - 2)$

$\therefore 3y + 3 = -x + 2$

$\therefore x + 3y + 1 = 0$

i. e. u = 0

$\therefore y + 1 = \frac{2}{3}(x - 2)$

$\therefore 3y + 3 = 2x - 4$

$\therefore 2x - 3y - 7 = 0$

i. e. v = 0

The joint equation of the required two lines is given by u .v = 0

$\therefore (x + 3y + 1)(2x - 3y - 7) = 0$

$\therefore 2x^2 + 3xy - 9y^2 - 5x - 24y - 7 = 0.$

This is the required joint equation.

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(ii) The given joint equation is $x^2 + xy - y^2 = 0$

Comparing with $ax^2 + 2hxy + by^2 = 0$, we get,

$a = 1 , 2h = 1, b = -1$

Let m_1 & m_2 be the slopes of the two lines.

$$m_1 + m_2 = \frac{-2h}{b} = \frac{-1}{-1} = 1 \qquad \text{and} \qquad m_1 \cdot m_2 = \frac{a}{b} = \frac{1}{-1} = -1$$

As the required lines are parallel to the given lines, $\therefore$ their slopes are m_1 & m_2.

As the lines are passing through the point (2 , - 3)

$\therefore$ the equations of lines by slope point form are:

$(y + 3) = m_1(x - 2)$

$\therefore m_1(x - 2) - (y + 3) = 0$

i. e. u = 0

$(y + 3) = m_2(x - 2)$

$\therefore m_2(x - 2) - (y + 3) = 0$

i. e. v = 0

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The joint equation of two lines is given by $u \cdot v = 0$

$\therefore [m_1(x-2)-(y+3)] \cdot [m_2(x-2)-(y+3)] = 0$

$\therefore (m_1 \cdot m_2)(x-2)^2 - (m_1 + m_2)(x-2)(y+3) + (y+3)^2 = 0$

$\therefore (-1)(x-2)^2 - (1)(x-2)(y+3) + (y+3)^2 = 0$

$\therefore -x^2 + 4x - 4 - xy - 3x + 2y + 6 + y^2 + 6y + 9 = 0$

$\therefore -x^2 - xy + y^2 + x + 8y + 11 = 0$

$\therefore x^2 + xy - y^2 - x - 8y - 11 = 0$

This is the required joint equation.

Ex. 6. Find the separate equation of the lines represented by the equation

$$(x-2)^2 - 3(x-2)(y+1) + 2(y+1)^2 = 0.$$

Solution:

The given equation is $(x-2)^2 - 3(x-2)(y+1) + 2(y+1)^2 = 0$

Let $x - 2 = X$ and $y + 1 = Y$,

$\therefore X^2 - 3XY + 2Y^2 = 0$

$\therefore X^2 - 2XY - XY + 2Y^2 = 0$

$\therefore X(X - 2Y) - Y(X - 2Y) = 0 \quad \Rightarrow \quad (X - 2Y)(X - Y) = 0$

This equation is of the form $u \cdot v = 0$, whose separate equation are $u = 0$ & $v = 0$,

$\therefore X - 2Y = 0$ & $X - Y = 0$

On back substituting the values of X & Y, we get

$\therefore x - 2 - 2y - 2 = 0$ & $x - 2 - y - 1 = 0$

$\therefore x - 2y - 4 = 0$ & $x - y - 3 = 0$ are the required separate equation of the lines.

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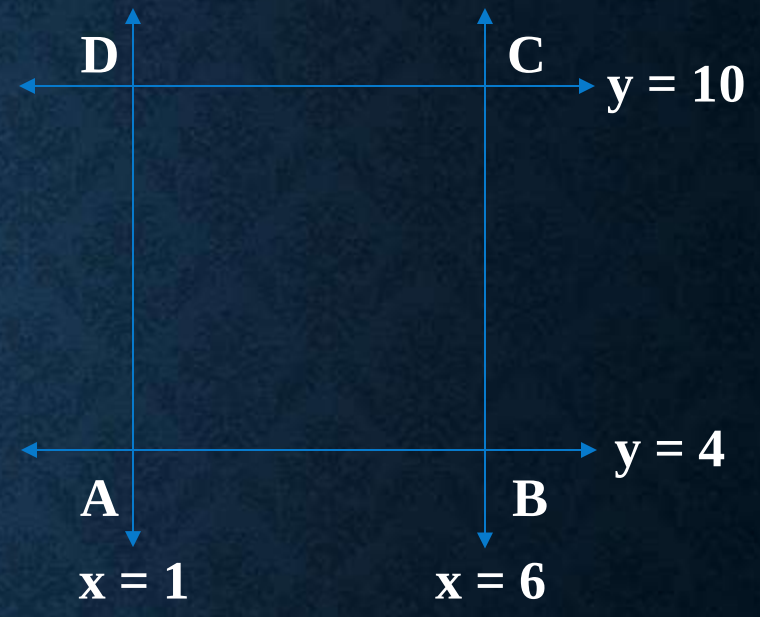
Ex. 7. Equations of pair of opposite sides of a parallelogram are $x^2 - 7x + 6 = 0$ and $y^2 - 14y + 40 = 0$. Find the joint equation of its diagonals.

Solution:

The given joint equations are $x^2 - 7x + 6 = 0$ and $y^2 - 14y + 40 = 0$



$x^2 - 7x + 6 = 0$

$\therefore (x - 1)(x - 6) = 0$

$\therefore x - 1 = 0$ & $x - 6 = 0$

$\therefore x = 1$ & $x = 6$

Lines are parallel to Y – axis.

$y^2 - 14y + 40 = 0$

$\therefore (y - 4)(y - 10) = 0$

$\therefore y - 4 = 0$ & $y - 10 = 0$

$\therefore y = 4$ & $y = 10$

Lines are parallel to X – axis.

This lines intersect each other at points A, B, C, D which forms a parallelogram.

Where A(1, 4), B(6, 4), C(6, 10) & D(1, 10)

$\therefore$ Equation of diagonals AC and BD are:

$$\frac{y - 4}{10 - 4} = \frac{x - 1}{6 - 1}$$

$\therefore 5y - 20 = 6x - 6$

$\therefore 6x - 5y + 14 = 0$

i. e. $u = 0$

$$\frac{y - 4}{10 - 4} = \frac{x - 6}{1 - 6}$$

$\therefore -5y + 20 = 6x - 36$

$\therefore 6x + 5y - 56 = 0$

i. e. $v = 0$

The joint equation of the lines is $u \cdot v = 0$

$(6x - 5y + 14)(6x + 5y - 56) = 0$

$\therefore 36x^2 - 25y^2 - 252x + 350y - 784 = 0$

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Ex. 8. Δ OAB is formed by the lines $x^2 - 4xy + y^2 = 0$ and the line $x + y - 2 = 0$. Find the equation of the median of the triangle drawn through O.

Solution:

Let the co-ordinates of A & B be (x_1, y_1) & (x_2, y_2) respectively.

The midpoint of segment AB is $P\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

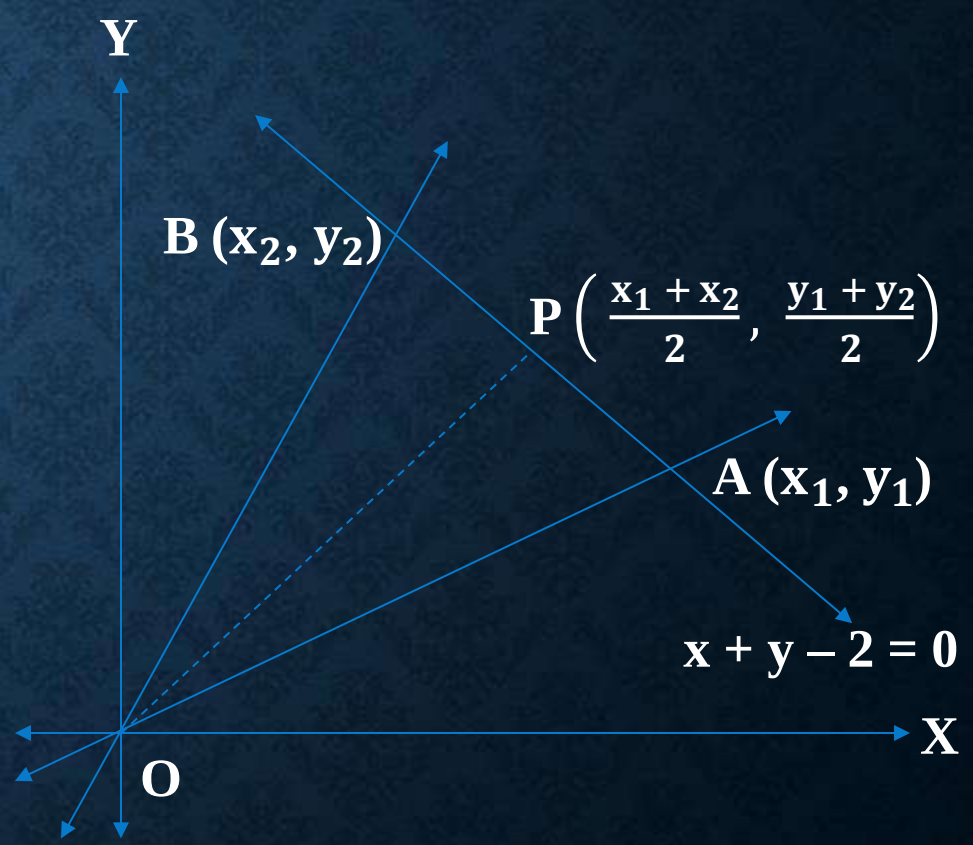
The co-ordinates of A & B can be obtained by solving equations $x + y - 2 = 0$ and $x^2 - 4xy + y^2 = 0$ simultaneously.

Put $y = 2 - x$ in $x^2 - 4xy + y^2 = 0$, we get,

$$x^2 - 4x(2 - x) + (2 - x)^2 = 0$$

$$\therefore x^2 - 8x + 4x^2 + 4 - 4x + x^2 = 0$$

$$\therefore 6x^2 - 12x + 4 = 0$$



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$\therefore 3x^2 - 6x + 2 = 0$

x_1 & x_2 are roots of this equation,

$\therefore x_1 + x_2 = -\frac{-6}{3} = 2 \quad \Rightarrow \quad \frac{x_1 + x_2}{2} = 1$

$\therefore$ The x co-ordinate of P is 1.

As P lies on the line $x + y - 2 = 0$

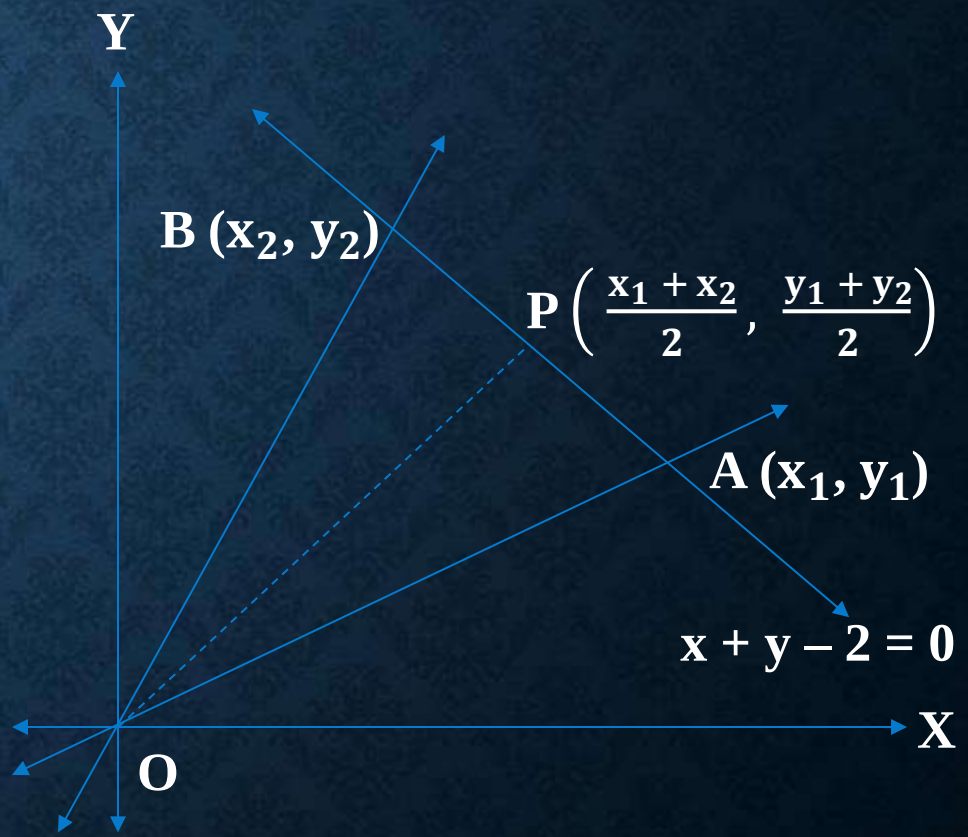
$\therefore 1 + y - 2 = 0 \quad \Rightarrow \quad y = 1$

$\therefore$ Co-ordinates of P are (1, 1).

$\therefore$ The equation of the median OP is: $\frac{y - 0}{1 - 0} = \frac{x - 0}{1 - 0}$

$\therefore y = x$

i. e. $x - y = 0$.



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